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# CAPACITY THEOREM OF H. BREZIS AND F.E. BROWDER TYPE IN WEIGHTED VARIABLE EXPONENT SOBOLEV SPACES AND APPLICATION TO THE UNILATERAL PROBLEM

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**Abstract.** In this paper we give and prove some properties of the capacity theory in the weighted variable exponent Sobolev spaces, we generalize the Theorem of H. Brezis and F.E. Browder in the setting of the weighted variable exponent Sobolev spaces, which extends the previous result of H. Brezis and F.E. Browder and we make an application to an unilateral problem.

**Keywords**: Weighted variable exponent sobolev spaces, capacity, Theorem of H. Brezis and F.E. Browder, unilateral problem.

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## 1 Introduction

The capacity plays a very important role in the study of the potential theory which becomes an extensive domain of research rich in ideas and methods applied in our time in mathematics and physics, particularly in function theory, functional analysis, probability theory, approximation theory and harmonic analysis.

The theory of capacity and non-linear potential in the classical Lebesgue space $L^p(\Omega)$, was mainly studied in Maz'ya & Khavin (1972) and Meyers (1970). These authors in their previous works have introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis and in the theory of partial differential equations.

This notion was generalized to Orlicz spaces by Aissaoui & Benkirane (1994), to Musielak-Orlicz spaces by Hassib et al. (2017) and to the weighted variable exponent Sobolev spaces by Ismail (2012). Thus, the first goal of this paper we gives and prove some proprieties of the $C_{p(.),\omega}$ capacity in the setting of the weighted variable exponent Sobolev spaces defined by Ismail (2012). As an application we generalize the theorem of Browder F.F by Browder F.E. in the setting of the weighted variable exponent Sobolev space, which extends the previous result of Brezis & Browder (1982), and we make an application to an unilateral problem.

# 2 Preliminary

## 2.1 Variable Exponent Lebesgue and Sobolev Spaces.

Let $\Omega$ be a bounded open subset of $I\!R^N$ $(N \geq 2)$,

$$C_+(\overline{\Omega}) = \{continuous \ \ function \ \ p(\cdot) : \overline{\Omega} \longrightarrow I\!R \ \ \text{such that} \ \ 1 < p_- \leqslant p(x) \leqslant p_+ \leqslant N\},$$

where

$$p_- = \min\{p(x) \,/\, x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = \max\{p(x) \,/\, x \in \overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \{u : \Omega \longrightarrow I\!R \ \ \text{measurable} \ / \ \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

The space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ (see Zhao et al (1997)).

**Proposition 1.** ***(see Zhikov (2004), Zhao et al (1997))***
*If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx \quad \forall u \in L^{p(x)}(\Omega),$$

*then, the following assertions holds*
***(i)*** $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1),$
***(ii)*** $\|u\|_{p(x)} > 1 \ \Rightarrow \ \|u\|^{p_-}_{p(x)} \leq \rho(u) \leq \|u\|^{p_+}_{p(x)}$ ,
***(iii)*** $\|u\|_{p(x)} < 1 \ \Rightarrow \ \|u\|^{p_+}_{p(x)} \leq \rho(u) \leq \|u\|^{p_-}_{p(x)},$
***(iv)*** $\|u_n\|_{p(x)} \to 0 \ \Leftrightarrow \ \rho(u_n) \to 0 \quad$ *and* $\quad \|u_n\|_{p(x)} \to \infty \ \Leftrightarrow \ \rho(u_n) \to \infty.$

Now, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{\, u \in L^{p(x)}(\Omega) \ \text{and} \ |\nabla u| \in L^{p(x)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

Where $\nabla u = (\dfrac{\partial u}{\partial x_1}; \dfrac{\partial u}{\partial x_2}; ........; \dfrac{\partial u}{\partial x_N}).$

## 2.2 The weighted variable exponents Lebesgue and Sobolev spaces.

Let $\Omega$ be a bounded open subset of $I\!R^N$, $N \geq 2$ , set

$$C_+(\overline{\Omega}) = \{p \in C(\Omega) : \min_{x \in \overline{\Omega}} p(x) \geqslant 1\},$$

For all $p \in C_+(\overline{\Omega})$, we define $p^+ = \sup\limits_{x \in \Omega} p(x)$ and $p_- = \inf\limits_{x \in \Omega} p(x)$.
We say that $p(.)$ is log-Hölder continuous in $\Omega$ if

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \qquad \forall x, y \in \overline{\Omega} \quad \text{such that} \quad |x-y| < \frac{1}{2},$$

We denote by $P^{log}(\Omega)$ the class of variable exponents which are log-Hölder continuous in $\Omega$.

Let $\omega$ be a function defined in $\Omega$, $\omega$ is called a weight function in $\Omega$ if she is measurable and strictly positive a.e. in $\Omega$.

Let $p \in C_+(\overline{\Omega})$ and $\omega$ be a weighted function in $\Omega$. The weighted variable exponents Lebesgue space $L^{p(x)}(\Omega, \omega)$, consists of all real valued functions such that

$$u : \Omega \to I\!R, measurable \;\; and \;\; \int_\Omega \omega(x)|u|^{p(x)}dx < \infty$$

ie

$$L^{p(x)}(\Omega, \omega) = \{u : \Omega \to I\!R, measurable : \int_\Omega \omega(x)|u|^{p(x)}dx < \infty\},$$

provided with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega,\omega)} = \inf\Big\{\mu > 0 : \int_\Omega \omega(x)|\frac{u(x)}{\mu}|^{p(x)}dx \le 1\Big\}$$

$L^{p(x)}(\Omega, \omega)$ is a reflexive Banach space.

**Lemma 1.** *Kovacik & Rakonsik (1991) For all function $u \in L^{p(x)}(\Omega, \omega)$, we denoted*

$$\rho_{\Omega,\omega}(u) = \int_\Omega \omega(x)|u|^{p(x)}dx.$$

*Then*

***(i)*** $\rho_{\Omega,\omega}(u) > 1 \quad (= 1; < 1) \Leftrightarrow \|u\|_{L^{p(x)}(\Omega,\omega)} > 1 \quad (= 1; < 1)$, *respectively.*

***(ii)*** *If* $\|u\|_{L^{p(x)}(\Omega,\omega)} > 1$ *then* $\|u\|^{p-}_{L^{p(x)}(\Omega,\omega)} \le \rho_{\Omega,\omega}(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)}$.

***(iii)*** *If* $\|u\|_{L^{p(x)}(\Omega,\omega)} < 1$ *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)} \le \rho_{\Omega,\omega}(u) \le \|u\|^{p-}_{L^{p(x)}(\Omega,\omega)}$.

We define the weighted variable exponents Sobolev space by

$$W^{1,p(x)}(\Omega, \omega) = \Big\{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega, \omega), i = 1, ..., N\Big\},$$

provided with the norme

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \|u\|_{L^{p(x)}(\Omega)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(\Omega,\omega)}$$

which is equivalent to the Luxemburg norme

$$|||u||| = \inf\Bigg\{\mu > 0 : \int_\Omega \Bigg(|\frac{u}{\mu}|^{p(x)} + \omega(x)\sum_{i=1}^N |\frac{\frac{\partial u}{\partial x_i}}{\mu}|^{p(x)}\Bigg)dx \le 1\Bigg\}.$$

Let $\omega$ a weight function such that the following conditions:

$(w_1)$ $\quad \omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$;

$(w_2)$ $\quad \omega^{-s(x)} \in L^1(\Omega, \omega)$ with $s(x) \in \big(\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x) - 1}, \infty\big)$.

**Remark 1.** *Kovacik & Rakonsik (1991)*

• *Let $\omega$ be a positive measurable and finite function such the condition $(w_1)$ holds. Then the space $\big(W^{1,p(x)}(\Omega, \omega), \|.\|_{W^{1,p(x)}(\Omega,\omega)}\big)$ is a reflexive Banach space.*

• *If $(w_1)$ holds, then $W^{1,p(x)}(\Omega, \omega)$is a reflexive Banach space and $C_0^\infty(\Omega) \subset W^{1,p(x)}(\Omega, \omega)$.*

• *The dual of the weighted Sobololev space $W_0^{1,p(x)}(\Omega, \omega)$ is equivalent to $W_0^{-1,p'(x)}(\Omega, \omega^*)$, where $\omega^* = \omega^{1-p'(x)}$ and $p'(x) = \dfrac{p(x)}{p(x) - 1}$.*

**Remark 2.** *Kovacik & Rakonsik (1991) If we set*

$$\rho^{1}_{(\Omega,\ \omega)}(u) = \int_{\Omega} |u|^{p(x)} \ + \omega(x)|\nabla u|^{p(x)} dx \quad \forall u \in W^{1,p(x)}(\Omega,\omega)$$

*We have*

$$min(\|u\|^{p-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p+}_{W^{1,p(x)}(\Omega,\omega)}) \leqslant \rho^{1}_{(\Omega,\ \omega)}(u) \leqslant max(\|u\|^{p-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p+}_{W^{1,p(x)}(\Omega,\omega)}).$$

**Definition 1.** *Diening & Hasto (2008) Let $p \in C_{+}(\overline{\Omega})$. The class $A_{p(.)}$ consists of those weight $\omega$ for which*

$$\sup_{B\in \mathbf{B}} |B|^{-\mathbf{p}_B} ||\omega||_{L^1(B)} ||\frac{1}{\omega}||_{L^{\frac{p'(.)}{p(.)}}} < \infty,$$

*where $\mathbf{B}$ denotes the set of balls in $\mathbb{R}^N$ , $\mathbf{p}_B = (\frac{1}{|B|}\int_B \frac{1}{p(x)})^{-1}$*

**Theorem 1.** *Ismail (2012) Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, and $\omega \in A_{p(.)}$. Then $C_0^{\infty}(\mathbb{R}^N)$ is dense in $W^{1,p(x)}(\mathbb{R}^N,\omega)$.*

In the following $\omega$ is a positive measurable and finite function which satisfies the condition $(w_1)$.
The following Lemma can be proved similarly in Lemma 2.4 of Benkirane et al. (2013).

**Lemma 2.** *Let $u \in W_0^{1,p(x)}(\Omega,\omega)$. There existe a sequence $u_n$ such that:*
***(i)*** *$u_n \in W_0^{1,p(x)}(\Omega,\omega) \cap L^{\infty}(\Omega)$,*
***(ii)*** *supp $u_n$ is compact,*
***(iii)*** *$|u_n| \leqslant |u|$ a.e. in $\Omega$,*
***(iv)*** *$u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega,\omega)$,*
***(v)*** *$u_n u \geqslant 0$ a.e. in $\Omega$ .*

## 2.3 Capacity

**Definitions 1.** *Let $T$ the classe of Borel sets in $\mathbb{R}^N$, and a function $C : T \to [0,+\infty]$.*
*1) $C$ is called capacity if the following axioms are satisfied:*
*i) $C(\emptyset) = 0$.*
*ii) $X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all $X$ and $Y$ in $T$.*
*iii) For all sequences $(X_n) \subset T$:*

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

*2) $C$ is called outer capacity if for all $X \in T$ :*

$$C(X) = \inf\{C(O) : O \supset X, \quad O \ \ open\}.$$

*3) $C$ is called an interior capacity if for all $X \in T$ :*

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ compact\}.$$

*4) A property, that holds true except perhaps on a set of capacity zero, is said to be true C-quasi-everywhere, ( abbreviated C-q.e).*
*5) $f$ and $(f_n)$ are real-valued finite functions C-q.e. We say that $(f_n)$ converges to $f$ in C-capacity if:*

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} C(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

*6) $f$ and $(f_n)$ are real-valued finite functions $C$-q.e. We say that $(f_n)$ converges to $f$ $C$-quasi-uniformly, (abbreviated $C$-q.u) if :*

$$(\forall \varepsilon > 0), (\exists\ X \in T)\ :\ C(X) < \varepsilon \ \text{ and } (f_n) \text{ converges to } f \text{ uniformly on } X^c.$$

**Propositions 1.** *Ismail (2012) For $E \subset \mathbb{R}^N$, we denote*

$$S_{p(.),\omega}(E) = \{u \in W^{1,p(x)}(\mathbb{R}^N, \omega)\ :\ u \geqslant 1 \text{ on an open set containing } E\}.$$

*The Sobolev $(p(.), \omega)$-capacity of $E$ is defined by*

$$C_{p(.),\omega}(E) = \inf_{u \in S_{p(.),\omega}(E)} \rho_{1,p(.),\omega}(u) = \inf_{u \in S_{p(.),\omega}(E)} \int_\Omega |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \omega(x) dx.$$

*In case $S_{p(.),\omega}(E) = \varnothing$, we set $C_{p(.),\omega}(E) = \infty$. The $C_{p(.),\omega}$-capacity has the following proprieties.*
*(i) $C_{p(.),\omega}(\emptyset) = 0$.*
*(ii) $X \subset Y \Rightarrow C_{p(.),\omega}(X) \leqslant C_{p(.),\omega}(Y)$,*
*(iii) For all sequences $X_n \subset \mathbb{R}^N$:*

$$C_{p(.),\omega}(\bigcup_n X_n) \leqslant \sum_n C_{p(.),\omega}(X_n). \tag{1}$$

*(i∨) For all $X \subset \mathbb{R}^N$ :*

$$C_{p(.),\omega}(X) = \inf\{C_{p(.),\omega}(O) : O \supset X, \quad O \text{ open}\}. \tag{2}$$

*(∨) For all $X \subset \mathbb{R}^N$*

$$C_{p(.),\omega}(X) = \sup\{C_{p(.),\omega}(K) : K \subset X, \quad K \text{ compact}\}. \tag{3}$$

*(∨i) If there exists $f \in W^{1,p(x)}(\mathbb{R}^N, \omega)$ such that $f = +\infty$ on $E$ then,*

$$C_{p(.),\omega}(E) = 0 \tag{4}$$

*(∨ii) If $(X_n)$ is an increasing sequence of sets and $X = \bigcup_n X_n$, then*

$$\lim_{n \to +\infty} C_{p(.),\omega}(X_n) = C_{p(.),\omega}(X). \tag{5}$$

*(∨iii)If $X$ and $Y$ are subset of $\mathbb{R}^N$, then*

$$C_{p(.),\omega}(X \cup Y) + C_{p(.),\omega}(X \cap Y) \leqslant C_{p(.),\omega}(X) + C_{p(.),\omega}(Y). \tag{6}$$

**Lemma 3.** *Ismail (2012)*
*If $\omega(x) \geqslant 1$ for all $x \in \mathbb{R}^N$, then every measurable set $X \subset \mathbb{R}^N$ satisfies*

$$|X| \leqslant C_{p(.),\omega}(X), \tag{7}$$

*where $|X|$ is the lebesgue measure of $X$.*

# 3 The Main Results

## 3.1 Some proprieties of the $C_{p(.),\omega}$-capacity.

**Theorem 2.** *Let's consider the following propositions:*
*i)* $f_n \longrightarrow f$ *in* $W^{1,p(x)}(\mathbb{R}^N, \omega)$.
*ii)* $f_n \longrightarrow f$ *in* $C_{p(.),\omega} -$ *capacity.*
*iii) There exists a subsequence* $(f_{n_j})$ *such that :* $f_{n_j} \longrightarrow f, \quad C_{p(.),\omega} - q.u.$
*iv) There exists a subsequence* $(f_{n_j})$ *such that* $f_{n_j} \longrightarrow f, \quad C_{p(.),\omega} - q.e.$
*We have* $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$

*Proof.* Let show that $i) \Rightarrow ii)$.
By (4 ) we have $f$ and $f_n$ are finite for every n; $C_{p(.),\omega} - q.e.$
Let $\varepsilon > 0$, we have

$$C_{p(.),\omega}(\{x \ : \ |f_n - f|(x) > \varepsilon\}) \leqslant \rho_{1,p(.),\omega}(\frac{f_n - f}{\varepsilon}).$$

Since $f_n \longrightarrow f$ *in* $W^{1,p(x)}(\mathbb{R}^N, \omega)$, then

$$(\forall \varepsilon > 0): \quad \rho_{1,p(.),\omega}(\frac{f_n - f}{\varepsilon}) \longrightarrow 0.$$

Therefore,

$$\lim_{n \to +\infty} C_{p(.),\omega}(\{x \ : \ |f_n - f|(x) > \varepsilon\}) = 0.$$

Let show that $ii) \Rightarrow iii)$.
Let $\varepsilon > 0$ $\exists f_{n_j}$ such that $C_{p(.),\omega}(\{x \ : \ |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}$.
We put

$$E_j = \{x \ : \ |f_{n_j} - f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j,$$

we have $C_{p(.),\omega}(G_m) \leqslant \sum_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.
On the other hand,

$$(\forall x \in (G_m)^c) \ : \ |f_{n_j} - f|(x) \leqslant 2^{-j}, (\forall j \geqslant m).$$

Thus

$$f_{n_j} \longrightarrow f \quad C_{p(.),\omega} - q.u.$$

Let show that $iii) \Rightarrow iv)$.
We have $\forall j \in \mathbb{N}, \exists X_j : C_{p(.),\omega}(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \longrightarrow f$ *on* $(X_j)^c$.
We put $X = \bigcap_j X_j$, then $C_{p(.),\omega}(X) = 0$ *and* $f_{n_j} \longrightarrow f$ *on* $X^c$. □

**Theorem 3.** *Let* $\omega$ *a positive measurable and finite function, such the condition* $(w_1)$ *holds. If* $f_n$ , $f \in W^{1,p(x)}(\mathbb{R}^N, \omega)$ *such that* $f_n \rightharpoonup f$ *weakly in* $W^{1,p(x)}(\mathbb{R}^N, \omega)$, *then*

$$\liminf(f_n)(x) \leqslant f(x) \leqslant \limsup f_n(x) \quad C_{p(.),\omega} - q.e.$$

*Proof.* $W^{1,p(x)}(\mathbb{R}^N, \omega)$ is reflexive. By the Banach-Saks theorem, there is a subsequence denoted again $(f_n)$ such that the sequence; $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converge to f strongly in $W^{1,p(x)}(\mathbb{R}^N, \omega)$. By theorem 2, there is a subsequence of $(g_n)$ denoted again $(g_n)$ such that

$$\lim_{n \to +\infty} g_n(x) = f(x) \quad C_{p(.),\omega} - q.e.$$

On the other hand,

$$\liminf f_n(x) \leqslant \lim_{n\to+\infty} g_n(x) \quad .$$

Therefore,

$$\liminf_{n\to+\infty} f_n(x) \leqslant f(x) \quad C_{p(.),\omega} - q.e.$$

For the second inequality, it suffices to replace $f_n$ by $(-f_n)$ in the first inequality. □

**Theorem 4.** *Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, and $\omega \in A_{p(.)}$.*
*For each $f \in W^{1,p(x)}(\mathbb{R}^N,\omega)$, there is a $C_{p(.),\omega}$-quasicontinuous function $g \in W^{1,p(x)}(\mathbb{R}^N,\omega)$ such that $f = g$ $C_{p(.),\omega}$.q.e. The function $g$ is called a $C_{p(.),\omega}$-quasicontinuous representative of the functionf.*

*Proof.* Let $f \in W^{1,p(x)}(\mathbb{R}^N,\omega)$. By theorem 1, there exists a sequence $(f_n)$ in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in $W^{1,p(x)}(\mathbb{R}^N,\omega)$.
By theorem 2, there exists a subsequence of $(f_n)$ denoted again by $(f_n)$ such that $f_n \longrightarrow f \;\; C_\varphi - q.u$, and the proof is complete. □

**Theorem 5.** *Let $1 < p^- \leqslant p^+ < \infty$, we have*
*1) If $O$ is an open set of $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ such that $|E| = 0$, then*

$$C_{p(.),\omega}(O) = C_{p(.),\omega}(O \setminus E).$$

*2) $u$ and $v$ are $C_{p(.),\omega}$-quasicontinuous functions in $\mathbb{R}^N$, we have*

*i) if $u = v$, almost everywhere in an open set $O \subset \mathbb{R}^N$ then $u = v$ $C_{p(.),\omega}-$quasieverywhere in $O$,*

*ii) if $u \leqslant v$, almost everywhere in an open set $O \subset \mathbb{R}^N$ then $u \leqslant v$ $C_{p(.),\omega}-$quasieverywhere in $O$.*

*Proof.* 1) It obvious that $C_{p(.),\omega}(O) \geqslant C_{p(.),\omega}(O \setminus E)$. Let $u \in S_{p(.),\omega}(O \setminus E)$ thus $u \geqslant 1$ in an open containing $O \setminus E$. Let the function $f$ define as

$$\begin{cases} f(x) = u(x) & , if \ x \in \ \mathbb{R}^N \setminus E \\ f(x) = 1 & , if \ x \in E. \end{cases}$$

We have $f \in S_{p(.),\omega}(O)$ and $\rho_{1,p(.),\omega}(f) = \rho_{1,p(.),\omega}(u)$, thus

$$C_{p(.),\omega}(O) \leqslant \rho_{1,p(.),\omega}(u),$$

and by passing to inf we get $C_{p(.),\omega}(O) \leqslant C_{p(.),\omega}(O \setminus E)$.
2) Since $C_{p(.),\omega}$ is an outer capacity we get the results by Kilpeläinen (1998). □

**Theorem 6.** *Let $w(x) \geq 1$ for $x \in \mathbb{R}^N$. If $(f_n)_n$ is a sequence which converge to $f$ in $W^{1,p(.)}(\mathbb{R}^n,\omega)$, then there exists a subsequence of $(f_n)_n$ which converge to $f$ q.e and a.e.*

*Proof.* If $f_n \longrightarrow f$ *in* $W^{1,p(x)}(\mathbb{R}^N,\omega)$, then by theorem 2 there exists a subsequence $(f_{n_j})$ such that $f_{n_j} \longrightarrow f, \quad C_{p(.),\omega} - q.e.$
Thus there exists a measurable subset $E$ of $\mathbb{R}^N$ such that $f_{n_j} \longrightarrow f$ in $E^c$ and $C_{p(.),\omega}(E) = 0$.
By Lemma 3 we have $|E| \leqslant C_{p(.),\omega}(E)$, therefore $f_{n_j} \longrightarrow f$ a.e. □

**Lemma 4.** *Let $\Omega$ be a open subset of $R^N$. and $T \in W^{-1,p'(x)}(\Omega,\omega^*) \cap M(\Omega)$, where $M(\Omega)$ denote the set of Radon measures in $\Omega$.*
*If $X \subset \Omega$ is such that $C_{p(.),\omega}(X) = 0$, then $X$ is $|T|$ -measurable and $|T|(X) = 0$.*

*Proof.* The same as in GRUN-Rehomme (1977) and Brezis & Browder (1982). □

### 3.2 A theorem of H. Brezis and F. Brower type in weighted variable exponents Sobolev space

Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, $\omega(x) \geq 1$ and $\omega \in A_{p(.)}$.
In this section, we generalize the theorem of H Brezis and F.E. Browder Brezis & Browder (1982) in the setting of the weighted variable exponents Sobolev space $W^{1,p(x)}(\Omega, \omega)$.
Let $\Omega$ be a open subset of $\mathbb{R}^N$. In this section we study the following question: let $u \in W_0^{1,p(x)}(\Omega, \omega)$ and $T \in W^{-1,p'(x)}(\Omega, \omega^*)$ such that $T = \mu + h$, where $\mu$ lies in $M^+(\Omega)$ (the subset of positive Radon measures) and h in $L^1_{loc}(\Omega)$; find sufficient conditions on the data in order for $u$ to belong $L^1(\Omega; d\mu)$, for $hu$ to belong to $L^1(\Omega)$ and finally to have:

$$< T, u >= \int_\Omega u d\mu + \int_\Omega hu dx.$$

This question was solved in Boccardo et al. (1990) in the case of the classical Sobolev spaces, in Benkirane & Gossez (1994) when $\mu = 0$ in the case of Orlicz Sobolev spaces, in Benkirane (1986) in the case of Orlicz Sobolev spaces and in Hassib et al. (2017) in the case of Musielak-Orlicz Sobolev spaces.

**Theorem 7.** *Let $\Omega$ be a open subset of $\mathbb{R}^N$. Consider $u \in W_0^{1,p(x)}(\Omega, \omega)$, $u \geqslant 0$ a.e in $\Omega$ and $T \in W^{-1,p'(x)}(\Omega, \omega^*)$ such that $T = \mu + h$, where $\mu$ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and $h \in L^1_{loc}(\Omega)$, assume that:*

$$hu \geqslant -|\Phi| \quad a.e \ in \ \Omega \ for \ some \ \ \Phi \ in \ L^1(\Omega). \tag{8}$$

*Then:*

$$hu \in L^1(\Omega), u \in L^1(\Omega; d\mu) \ and \ < T, u >= \int_\Omega u d\mu + \int_\Omega hu dx. \tag{9}$$

**Remark 3.** *Note that $\mu(X) = 0$ for all $X \subset \Omega$ such that $C_{p(.),\omega}(X) = 0$.*
*Indeed by lemma 4*

$$|T|(X) = |\mu + h|(X) = 0,$$

*but*

$$0 \leqslant \mu(X) \leqslant |h|(X) + |\mu + h|(X) = 0.$$

Let prove the theorem 7.

*Proof.* Let $u \in W_0^{1,p(x)}(\Omega, \omega)$, by Lemma 2 there exists a sequence $u_n$ such that:
**(i)** $u_n \in W_0^{1,p(x)}(\Omega, \omega) \cap L^\infty(\Omega)$,
**(ii)** supp $u_n$ is compact,
**(iii)** $|u_n| \leqslant |u|$ a.e. in $\Omega$,
**(v)** $u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega, \omega)$.
**(vi)** $u_n u \geqslant 0$ a.e. in $\Omega$ .
Following the lines of Boccardo et al. (1990), it is easy to deduce that

$$< \mu + h, u_n >= \int_\Omega u_n d\mu + \int_\Omega hu_n dx \tag{10}$$

Since $u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega, \omega)$, by using the theorem6, Lemma4 and remark3 we have

$$u_n \longrightarrow u \quad \mu.a.e \ \ and \ \ a.e. \ \ in \ \ \Omega. \tag{11}$$

We recall that by theorem 5, for any $v \in W_0^{1,p(x)}(\Omega,\omega)$ one has

$$v \geqslant 0 \quad a.e. \ \ in \ \Omega \Leftrightarrow v \geqslant 0 \quad q.e. \ \ in \ \Omega.$$

This equivalence, remark 3 and the fact ($u \geqslant 0$ a.e. in $\Omega$), imply

$$u_n \geqslant 0 \quad a.e. \ ; \ \ u_n \geqslant 0 \quad \mu.a.e. \ ; \quad 0 \leqslant u_n \leqslant u \ \ a.e. \ \ and \ \ 0 \leqslant u_n \leqslant u \quad \mu.a.e. \ \ in \ \ \Omega. \quad (12)$$

On the other hand, from $hu \geqslant -|\Phi|$ and $0 \leqslant u_n \leqslant u \ \ a.e. \ \ in \ \ \Omega$ we have

$$hu_n \geqslant -|\Phi| \ a.e.in \ \ \Omega \quad (13)$$

Since $< \mu + h, u_n >$ is bounded, (10 )and (12) imply $\displaystyle\int_\Omega hu_n dx \leqslant cst$; Similary (10 ) and (13 ) imply $\displaystyle\int_\Omega u_n d\mu \leqslant cst.$

By using (11), (12), (13) and Fatou's lemma we get $hu \in L^1(\Omega)$ and $u \in L^1(\Omega; d\mu)$.

Using $0 \leqslant u_n \leqslant u \quad \mu.a.e. \ \ in \ \ \Omega$ and $|hu_n| \leqslant |hu|$ a.e. in $\Omega$, it is now easy to pass to the limit in (10); we use the convergence of $u_n$ to $u$ in $W_0^{1,p(x)}(\Omega,\omega)$ for the left hand side and Lebesgue's dominated convergence theorem in each term of the right hand side: we obtain

$$< T, u >= \int_\Omega u d\mu + \int_\Omega hu dx.$$

□

### 3.3 Application to unilateral problem

Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, $\omega(x) \geq 1$ and $\omega \in A_{p(.)}$.

Consider some right hand side $f \in W^{-1,p'(x)}(\Omega,\omega^*)$ and the set

$$K_\Phi = \{v \in W_0^{1,p(x)}(\Omega,\omega), \ v \geqslant \Phi \ a.e \ in \ \Omega\}.$$

Where the obstacle $\Phi$ belong to $W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$.

The set $K_\Phi$ is convex, indeed let $v_1$ and $v_1$ in $K_\Phi$ and $\lambda \in [0,1]$, we have $v_1 \geqslant \Phi$ and $v_2 \geqslant \Phi \ a.e \ in \ \Omega$ then $\lambda v_1 + (1-\lambda)v_1 \geqslant \Phi \ a.e \ in \ \Omega$, thus $\lambda v_1 + (1-\lambda)v_1 \in K_\Phi$.

Let a pseudo-monotone mapping $S$ from $W_0^{1,p(x)}(\Omega,\omega)$ into $W^{-1,p'(x)}(\Omega,\omega^*)$. which satisfies the following conditions:

**(1)** $S$ is continuous from each finite-dimensional subspace of $W_0^{1,p(x)}(\Omega,\omega)$ into $W^{-1,p'(x)}(\Omega,\omega^*)$ for the weak* topology.

**(2)** $S$ maps bounded sets into bounded sets.

**(3)** $S$ is coercive, i.e that for some $v_0$ in $K_\Phi \cap L^\infty(\Omega)$

$$\frac{< S(v), v - v_0 >}{||v||_{W_0^{1,p(x)}(\Omega,\omega)}} \longrightarrow +\infty \ \ as \ \ ||v||_{W_0^{1,p(x)}(\Omega,\omega)} \longrightarrow +\infty. \quad (14)$$

Consider finally a Carathéodory function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ witch satisfies :

**(4)** $s.g(x,s) \geqslant 0, \forall \ s \in \mathbb{R} \ \ and \ \ a.e \ \ in \ \ \Omega,$

**(5)** $h_t = sup_{|s| \leqslant t}|g(x,s)| \in L^1(\Omega), \ \forall t \ \geqslant 0.$

**Theorem 8.** *The variational inequality:*

$$u \in K_\Phi, \ g(.,u) \in L^1(\Omega), \ ug(.,u) \in L^1(\Omega)$$

$$< Su, v - u > + \int_\Omega g(.,u)(v-u)dx \geqslant \quad < f, v - u >, \ \forall v \in K_\Phi \cap L^\infty(\Omega)$$

*has at least one solution.*

*Proof.* In the following we denote by $C_1$, $C_2$ and $C_3$, positive constants.
**First part** *Approximation and a priori istimates.*
Define $g_n(x,s) = \begin{cases} \chi_{\Theta_n}(x)g(x,s) \quad if \quad |g(x,s)| \leqslant n, \\ \chi_{\Theta_n}(x)n\dfrac{g(x,s)}{|g(x,s)|} \quad if \quad |g(x,s)| > n, \end{cases}$
where $\chi_{\Theta_n}$ is the characteristic function of the set $\Theta_n = \{x \in \Omega : |x| \leqslant n\}$
By using the proposition 1 of Gossez & Mustonen (1987) we have the approximate problem

$$\begin{cases} u_n \in K_\Phi, \\ < Su_n, v - u_n > + \displaystyle\int_\Omega g_n(., u_n)(v - u_n)dx \geqslant \; < f, v - u_n >, \; \forall v \in K_\Phi \cap L^\infty(\Omega) \end{cases} \tag{15}$$

has at least one solution.
Using $v = v_0$ as test function in (15) we get

$$< Su_n, u_n - v_0 > + \int_\Omega g_n(., u_n)(u_n - v_0)dx \leqslant \; < f, u_n - v_0 > . \tag{16}$$

If $(u_n)$ is not bonded in $W_0^{1,p(x)}(\Omega, \omega)$ then by the assumptions **(3)** we have

$$(\forall A > 0)(\exists n_0 \in \mathbb{N})(\forall n \geqslant n_o) : (\frac{< S(u_n), u_n - v_0 >}{||u_n||_{W_0^{1,p(x)}(\Omega,\omega)}} > A) \tag{17}$$

Let $E_n = \{x \in \Omega : u_n(x) \geqslant 0\}$, by (16) and (17) we have for large $n$ : $A||u_n||_{W_0^{1,p(x)}(\Omega,\omega)} + \int_{E_n} g_n(., u_n)(u_n - v_0)dx + \int_{\Omega - E_n} g_n(., u_n)u_n dx$
$\leqslant \int_{\Omega - E_n} g_n(., u_n)v_0 dx + ||f||_{W^{-1,p'(x)}(\Omega,\omega^*)}||u_n||_{W_0^{1,p(x)}(\Omega,\omega)} + ||f||_{W^{-1,p'(x)}(\Omega,\omega^*)}||v_0||_{W_0^{1,p(x)}(\Omega,\omega)}$
Let $G_n = \{x \in \Omega : u_n(x) \geqslant v_o(x)\}$ and $l = sup(|v_0|, |\Phi|)$.
By the assumption **(4)** and **(5)** we have

$$\int_{E_n \cap G_n} g_n(., u_n)(u_n - v_0)dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(., u_n)u_n dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(., u_n)v_0 dx \leqslant \int_\Omega |h_{||l||_{L^\infty(\Omega)}} v_0|,$$

$$\int_{\Omega - E_n} g_n(., u_n)u_n dx \geqslant 0,$$

$$\int_{\Omega - E_n} g_n(., u_n)v_0 dx \leqslant \int_\Omega |h_{||\Phi||_{L^\infty(\Omega)}} v_0|.$$

Then we get

$$||u_n||_{W_0^{1,p(x)}(\Omega)} \leqslant C_1, \; \forall n \geqslant n_0,$$

which is impossible, thus $(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$.
It follows that there exists a subsequence, again denoted by $u_n$ such that

$$u_n \rightharpoonup u, \;\; weakly \;\; in \;\; W_0^{1,p(x)}(\Omega) \;\; and \;\; a.e. \;\; in \;\; \Omega.$$

Thus

$$g_n(x, u_n(x)) \longrightarrow g(x, u(x)) \;\; a.e. \;\; in \;\; \Omega.$$

From (16) we get

$$\int_\Omega g_n(., u_n)(u_n - v_0)dx \leqslant C_2. \tag{18}$$

We shall prove

$$\int_\Omega |g_n(., u_n)(u_n - v_0)|dx \leqslant C_3.$$

Indeed

$$\begin{aligned}
\int_\Omega |g_n(., u_n)(u_n - v_0)|dx &= \int_{G_n \cap E_n} g_n(., u_n)(u_n - v_0)dx - \int_{G_n \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&- \int_{G_n^c \cap E_n} g_n(., u_n)(u_n - v_0)dx + \int_{G_n^c \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&\leqslant \int_\Omega g_n(., u_n)(u_n - v_0)dx - 2\int_{G_n \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&- 2\int_{G_n^c \cap E_n} g_n(., u_n)(u_n - v_0)dx \\
&\leqslant C_2 + 2\int_{G_n \cap E_n^c} g_n(., u_n)v_0dx + 2\int_{G_n^c \cap E_n} g_n(., u_n)v_0dx \\
&\leqslant C_2 + 4\int_\Omega |h_{||b||_{L^\infty}} v_0|dx = C_3,
\end{aligned} \tag{19}$$

where $b = sup(|\Phi|, |v_0|)$.
In order to prove

$$g_n(., u_n) \longrightarrow g(., u) \;\; in \;\; L^1(\Omega), \tag{20}$$

let us observe that, for any $\delta > 0$,

$$|g_n(x, u_n(x))| \leqslant \sup_{|t| \leqslant \delta^{-1} + ||v_0||_{L^\infty}} |g(., t)| + \delta |g_n(x, u_n(x))(u_n(x) - v_0(x))|,$$

and there fore, fore any measurable set $E$ in $\Omega$ we have

$$\int_E |g_n(., u_n)|dx \leqslant \int_E |h_{\frac{1}{\delta} + ||v_0||_{L^\infty}}| + \delta C_3.$$

By Vitali's theorem, we obtain (20).
Furthermore by (18) we have

$$\int_\Omega g_n(., u_n)u_n dx \leqslant C_2 + \int_\Omega g_n(., u_n)v_0 dx.$$

By Fatou's lemma and (20), we get

$$0 \leqslant \int_\Omega g(., u)u dx \leqslant C_2 + \int_\Omega g(., u)v_0 dx.$$

Thus

$$g(., u)u \in L^1(\Omega).$$

**Second part** : *Passing to the limit in (15)*
*Let*

$$\mu_n = Su_n - f + g_n(.,u_n).$$

*From (15) it is clear that* $\mu_n \in M^+(\Omega)$. *Since* $S$ *maps bounded sets of* $W_0^{1,p(x)}(\Omega,\omega)$ *in to bounded sets of* $W_0^{-1,p'(x)}(\Omega,\omega^*)$, *then we can assume for the same sequence that*

$$Su_n \rightharpoonup \chi \ \ weakly \ \ in \ W_0^{-1,p'(x)}(\Omega,\omega^*),$$

*which implies that*

$$\mu_n \longrightarrow \mu \ \ in \ \ D^{'}(\Omega),$$

*where*

$$\mu = \chi - f + g(.,u).$$

*We put* $\eta = u - \Phi, \ h = -g(.,u)$ *and* $T = \mu + h$.
*The assumptions of theorem 7 are satisfied since* $T = \chi - f \in W_0^{-1,p'(x)}(\Omega,\omega^*)$ *and* $h \in L^1(\Omega)$.
*Thus*

$$\begin{cases} u - \Phi \in L^1(\Omega; d\mu), \\ <\chi - f, u - \Phi >= \displaystyle\int_\Omega (u-\Phi)d\mu - \int_\Omega g(.,u)(u-\Phi)dx. \end{cases} \tag{21}$$

*Using* $v = \Phi$ *as test function in (15 ) we get*

$$< Su_n, u_n > \ \ \leqslant \ \ < Su_n, \Phi > - < f, \Phi - u_n > + \int_\Omega g_n(.,u_n)(\Phi - u_n),$$

*which gives passing to the limit and then using (21)*

$$\begin{cases} limsup_n < Su_n, u_n > \ \ \leqslant \ \ <\chi, \Phi > - < f, \Phi - u > + \displaystyle\int_\Omega g(.,u)(\Phi - u)dx, \\ \qquad\qquad\qquad\qquad\qquad \leqslant \ <\chi, u> \ \ + \ \ \displaystyle\int_\Omega (\Phi - u)d\mu \ \ \leqslant \ \ <\chi, u>; \end{cases} \tag{22}$$

*since, by theorem5 and remark 3 we have*

$$(\Phi - u) \leqslant 0 \quad \mu.a.e. \quad in \ \ \Omega. \tag{23}$$

*Using (22 ) and since* $S$ *is a pseudo - monotone operator, we obtain*

$$\chi = Su \quad and \quad < Su_n, u_n > \longrightarrow < Su, u > .$$

*It is now easy to pass to the limit in (15) for any fixed* $v \in K_\Phi \cap L^\infty(\Omega)$. □

## 4 Conclusion

In this work, we have stated and proved some properties of the capacity in the setting of the weighted variable exponent Sobolev spaces. As an application, we generalized the theorem of H Brezis and F.E. Browder in the setting of the weighted variable exponent Sobolev space, and we applied these results in the study of a unilateral problem.


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